Explicit static and dynamical solutions in chiral magnets without Heisenberg interaction

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The headlines

- The Landau-Lifshitz dynamics of a chiral magnet without Heisenberg interaction can be interpreted as a fluid flow equation.
- This fluid equation is particularly simple to solve when the chiral magnet has standard potential terms, i.e. Zeeman interaction + anisotropy.
- Looking for 'stationary flows', we find an infinite-dimensional space of static solutions.
- Specialising further to zero anisotropy, we find an infinite-dimensional space of dynamical solutions, modelling breathing/collapsing skyrmions.
- Static and dynamical solutions sometimes only continuous on a compact region.

Setting: the chiral magnet

$$E(\boldsymbol{n}) = \int \underbrace{\frac{1}{2}\partial_i \boldsymbol{n} \cdot \partial_i \boldsymbol{n}}_{Heisenberg} + \underbrace{\boldsymbol{A}_i \cdot (\boldsymbol{n} \times \partial_i \boldsymbol{n})}_{DMI} + \underbrace{h(1 - n_3)}_{Zeeman} + \underbrace{u(1 - n_3^2)}_{Anisotropy} d^2x$$

- $\boldsymbol{A}_i$ gives the preferred axis and strength of twisting of $\boldsymbol{n}$ along x_i
- ‘Axisymmetric DMI’ $\boldsymbol{A}_i = -kR(\beta)\boldsymbol{e}_i$, is most commonly considered.
- All β equivalent under rotation in spin space
- If $\beta = 0$: Bloch DMI $k\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n})$.

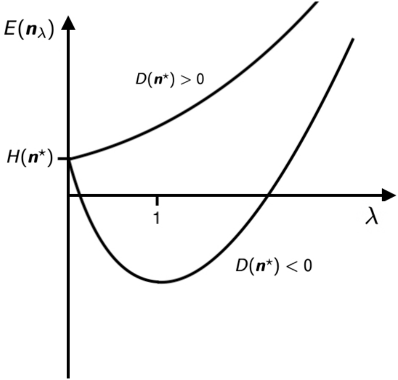
If potential enforces $\boldsymbol{n} =$ const. on boundary, no singularity formation, configurations classified by ‘topological charge’

$$Q(\boldsymbol{n}) = \frac{1}{4\pi} \int \boldsymbol{n} \cdot (\partial_1 \boldsymbol{n} \times \partial_2 \boldsymbol{n}) d^2x$$

Derrick scaling: why chirality allows for localised solutions

$$E(\mathbf{n}) = \int \underbrace{\frac{1}{2}\partial_i \mathbf{n}\cdot\partial_i \mathbf{n}}_{Heisenberg} + \underbrace{k\mathbf{n}\cdot(\nabla\times\mathbf{n})}_{DMI} + \underbrace{h(1-n_3)}_{Zeeman} + \underbrace{u(1-n_3^2)}_{Anisotropy} d^2x$$

- $\mathbf{n}^\star(\vec{x})$ minimiser of $E(\mathbf{n})$
- $E(\mathbf{n}^\star) = E_2(\mathbf{n}^\star) + E_1(\mathbf{n}^\star) + E_0(\mathbf{n}^\star)$
- $\mathbf{n}_\lambda(\vec{x}) = \mathbf{n}^\star\left(\frac{\vec{x}}{\lambda}\right)$
- $E(\mathbf{n}_\lambda) = E_2(\mathbf{n}^\star) + \lambda E_1(\mathbf{n}^\star) + \lambda^2 E_0(\mathbf{n}^\star)$
- $E_1(\mathbf{n}^\star) < 0$ for stability
- Finite barrier to collapse



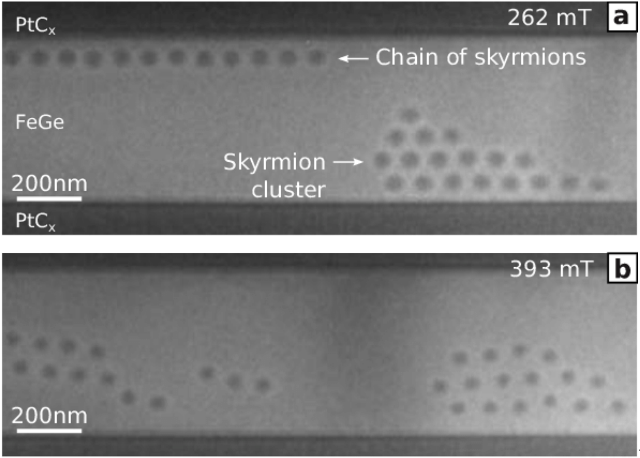
Setting: Landau-Lifshitz equation

Semiclassical approximation to evolution of spins:

$$\frac{d}{dt}\boldsymbol{n} = \boldsymbol{n} \times \frac{\delta E}{\delta \boldsymbol{n}} + \alpha \boldsymbol{n} \times \left(\boldsymbol{n} \times \frac{\delta E}{\delta \boldsymbol{n}} \right)$$

- Precession around local 'effective magnetic field' $H = -\frac{\delta E}{\delta \boldsymbol{n}}$, combined with relaxation towards H
- β is ratio of relaxation time to precession time:
  - $\alpha = 0$ conservative; Hamiltonian flow,
  - $\alpha \to \infty$ overdamped; gradient flow.
- In this presentation, we consider $\alpha = 0$.

Setting: Magnetic skyrmions



[1]

[1] Du et al. 2018.

Review: exact static minimisers

Axisymmetric ($O(2)$) solution in term of (r, ϕ):

$$\boldsymbol{n} = \begin{pmatrix} \sin\Theta(r)\cos(\phi+\gamma) \\ \sin\Theta(r)\sin(\phi+\gamma) \\ \cos\Theta(r) \end{pmatrix} \qquad (1)$$

Profile Θ, helicity γ

- $h = 0$, $u = 0$, $k = 0$: Belavin-Polyakov scale-free solution $\Theta = 2\arctan(\lambda/r)$, γ free
- when $u < 0$, $h = -2u$, Belavin-Polyakov solution with fixed scale $\lambda \sim k/h$ and helicity $\gamma = \pi/2$

We can relax symmetry to $U(1)$; then $\gamma(r)$. But also, completely symmetry-breaking solutions possible:

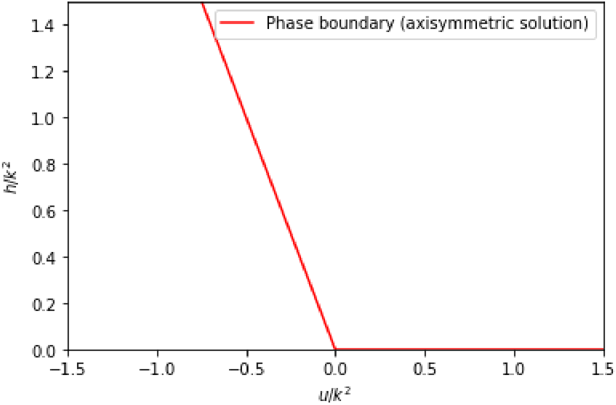
- when $u = -\frac{1}{2}k^2$, $h = k^2$, finite-dimensional moduli space of explicit solutions for any $Q(\boldsymbol{n}) \geq -1$ (in terms of stereographic projection)[2]

[2] Barton-Singer, Ross, and Schroers 2018.

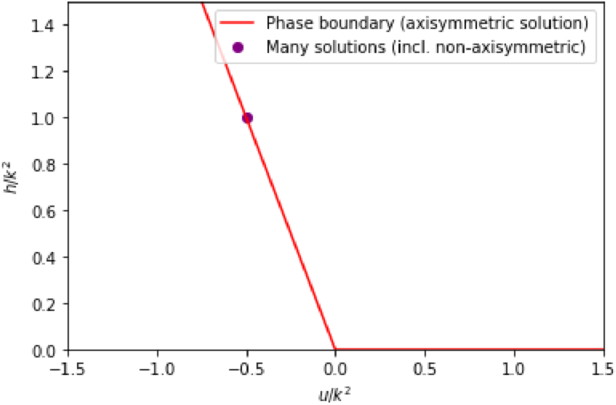
Review: existence of minimisers

- **Melcher 2014:** For $u = 0$, $h > k^2$, the minimum of $E(\boldsymbol{n})$ within the space of charge $Q(\boldsymbol{n}) = -1$ configurations is attained; Skyrmions exist.
- **Li, Melcher 2018:** For $u = 0$, h sufficiently large, there is a unique solution to the radial ODE, which is linearly stable wrt any (incl. non-radial) variations.
- **Gustafson, Wang 2021:** For $h + 2u = 1$, sufficiently small k depending on u, the solution to the radial ODE is unique and as $k \to 0$ 'close' to the Belavin-Polyakov solution ($k = h = u = 0$).

Review: Phase diagram



Review: Phase diagram



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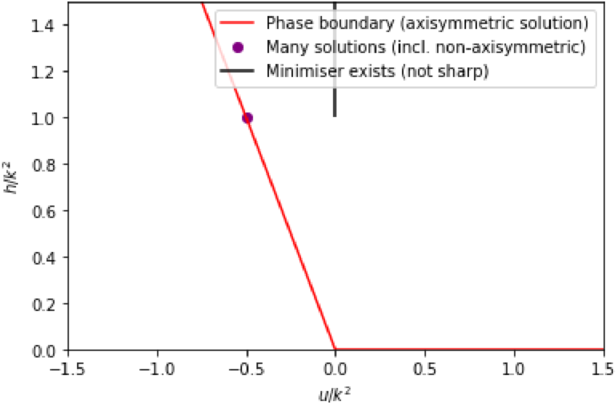
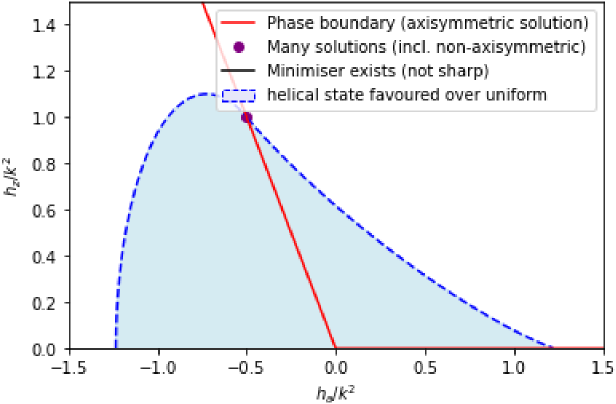
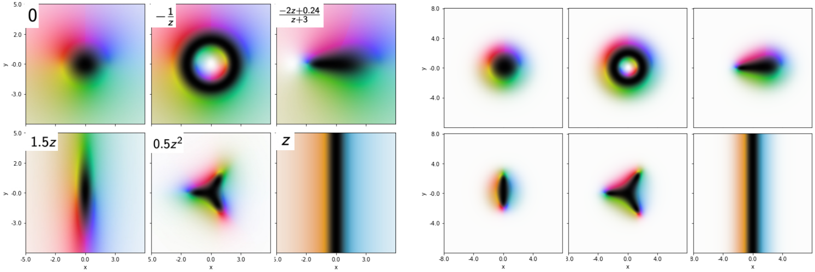


Review: Phase diagram



Example: the usefulness of exact solutions



Exact minimisers at critical point on left, minimisers away from the critical point (but still close to the transition to the helical state) on the right.

Specialised model: the chiral magnet with vanishing Heisenberg interaction

$$E(\boldsymbol{n}) = \int \underbrace{k\boldsymbol{n}\cdot(\nabla\times\boldsymbol{n})}_{DMI} + \underbrace{h(1-n_3)}_{Zeeman} + \underbrace{u(1-n_3^2)}_{Anisotropy}\, d^2x$$

- By spatial rescaling and rotation, *any* DMI where $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ not collinear (anisotropic, Bloch, Néel) can be brought into axisymmetric Bloch form above
- Any potential $f(n_3)$ works, in general we focus on the physically motivated form above.
- Scaling balance still exists, although solutions will be less smooth - inspiration from a similar limit in nuclear Skyrme model[3]

[3] Adam, Sánchez-Guillén, and Wereszczyński 2010.

Dynamics of the chiral magnet with vanishing Heisenberg interaction

Functional derivative of energy:

$$\frac{\delta E}{\delta \boldsymbol{n}} = 2k\nabla \times \boldsymbol{n} + f'(n_3)\boldsymbol{e}_3$$

Undamped Landau-Lifshitz:

$$\frac{d}{dt}\boldsymbol{n} = \boldsymbol{n} \times \frac{\delta E}{\delta \boldsymbol{n}} = -2k(\boldsymbol{n} \cdot \nabla)\boldsymbol{n} + f'(n_3)\boldsymbol{n} \times \boldsymbol{e}_3$$

Vector in plane $\vec{\nu} = (n_1, n_2)$ is like a fluid flow that transports the spins, i.e. if we define 'material derivative' $D_t = d_t + 2k\vec{\nu} \cdot \nabla$, then

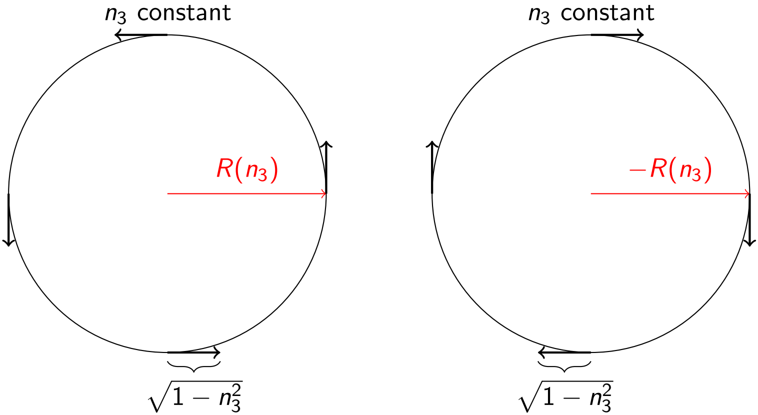
$$D_t\boldsymbol{n} = -f'(n_3)\boldsymbol{e}_3 \times \boldsymbol{n}$$

Thus: n_3 preserved along pathlines, $\vec{\nu}$ precesses along pathline at constant speed set by $f'(n_3)$.

Characteristics of this equation

Pathlines are circles (lines, points) with signed radius depending on n_3:

$$R(n_3) = \frac{-2k\sqrt{1-n_3^2}}{f'(n_3)}\ .$$



Static solutions example 1: uniform state

- Static solution $\leftrightarrow$ stationary fluid flow. Our pathlines become streamlines.
- If we specify the value of $\boldsymbol{n}$ at one point for static solution, immediately value of $\boldsymbol{n}$ is fixed along streamline.
- We can aim to construct static solutions by specifying these streamlines so they do not intersect.

Trivial case:

For any n_3^0 such that $f'(n_3^0) = 0$, we have straight streamline or point where $\boldsymbol{n}$ is constant. We can fill the plane with these to get $\boldsymbol{n} = (\sqrt{1 - n_3^{02}}, 0, n_3^0)$ (wlog).

Polarised and tilted uniform states reproduced

Example 2: axisymmetric skyrmions

We attempt to assemble a solution from concentric circular streamlines using the formula:

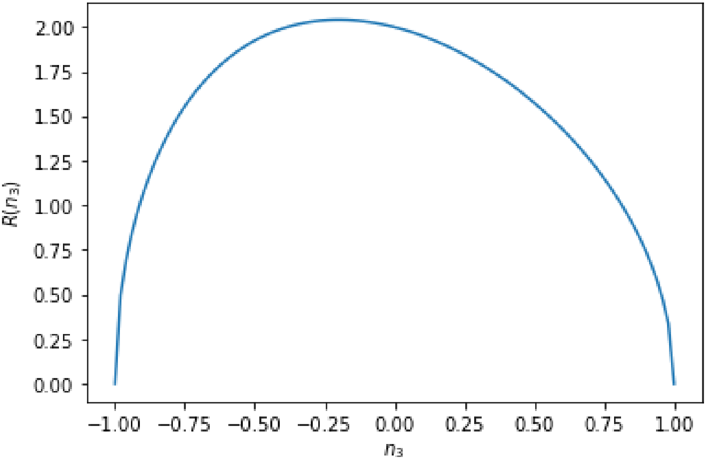
$$R(n_3) = \frac{2k\sqrt{1-n_3^2}}{h+2un_3}$$

Recipe for axisymmetric skyrmions

- Find range of n_3 where $R(n_3)$ monotonic and touches zero.
- Inverting the relation $R(n_3)$ on this interval gives a function $n_3(r)$ on a disc, or whole plane if $R(n_3)$ has a pole.
- Helicity $\gamma = \pm\frac{\pi}{2}$, depending on sign of $R(n_3)$.
- Can get discontinuous solution in whole plane by putting uniform background in rest of plane. Characteristics do not intersect.
- NB: since they are static solutions, they will also be valid for non-zero damping

Axisymmetric skyrmions: examples

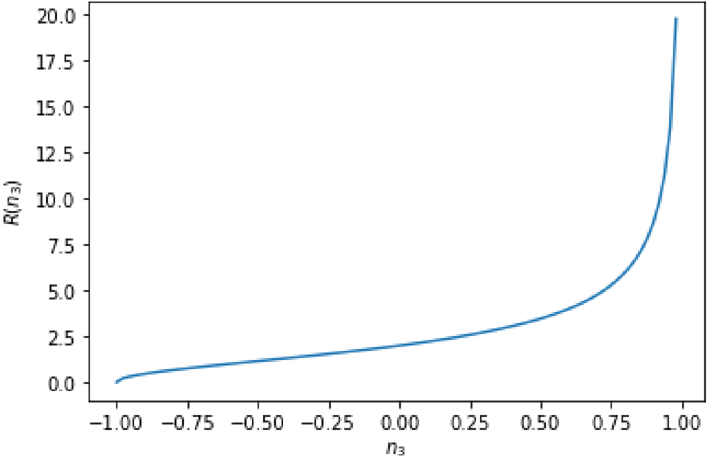
Case 1: $h - 2|u| > 0$



Two fractional-charge ($Q = \frac{u}{h} \pm \frac{1}{2}$) solutions inside disc of same finite size.

Axisymmetric skyrmions: examples

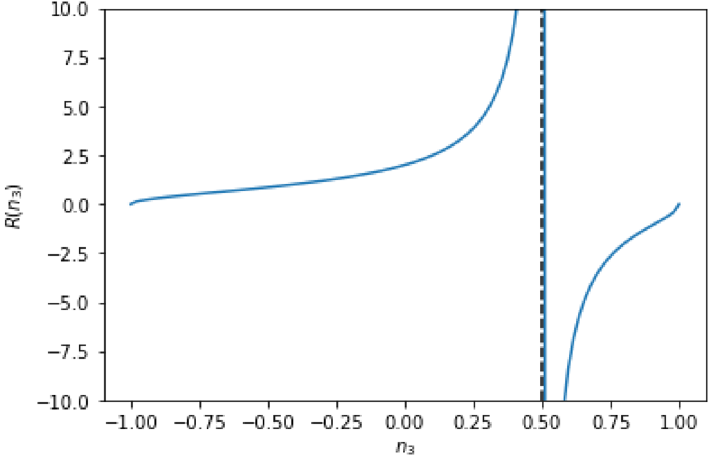
Case 2: $h + 2u = 0$



One $Q = -1$ solution on whole plane

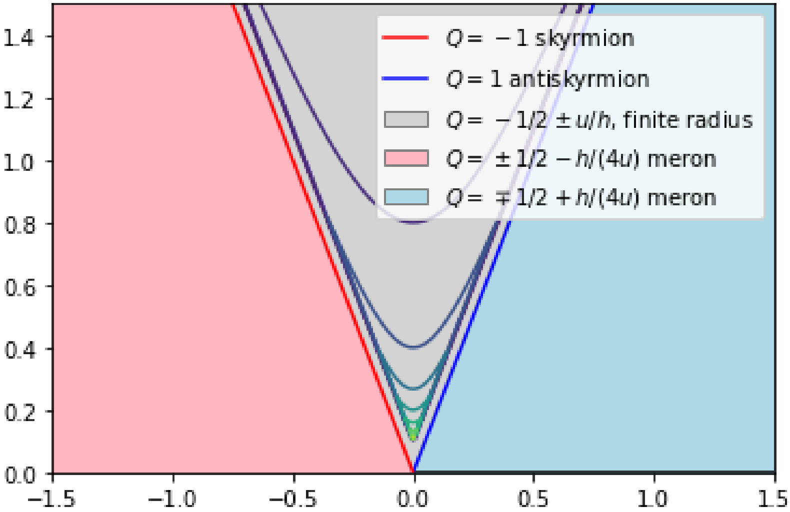
Axisymmetric skyrmions: examples

Case 3: $h + 2u < 0$

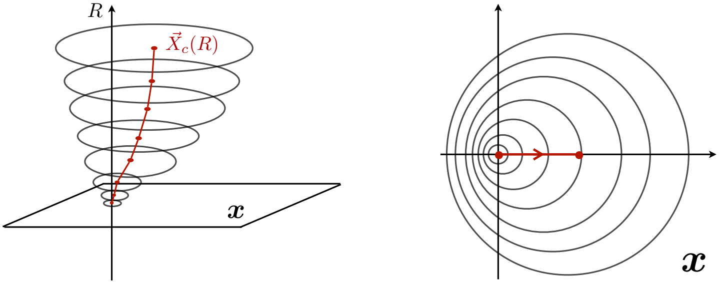


Two fractional-charge ($Q = \pm\frac{1}{2} - \frac{h}{4u}$, $\gamma = \pm\pi/2$) solutions covering plane.

Phase diagram revisited



Example 3: Non-axisymmetric solutions



- All of the solutions above (skyrmions, merons etc.) have degrees of freedom we have not explored.
- The pathline circles need not be concentric, provided circles of different n_3 do not intersect, and the domain is covered by circles.
- This translates to choosing a 1-Lipschitz function $\vec{X}_c(R)$ - infinite-dimensional moduli space of static continuous solutions

Dynamical solutions

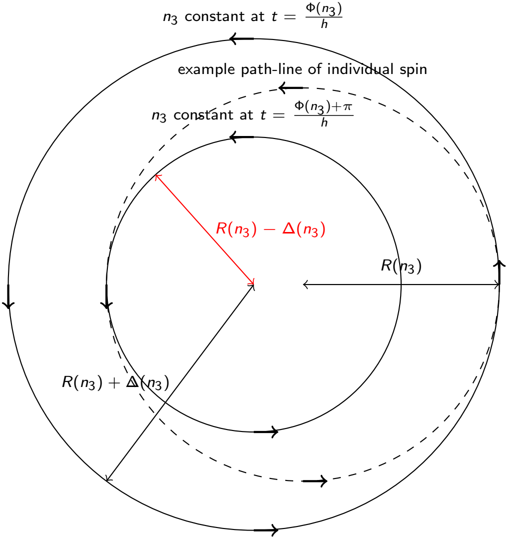
- We found the general pathline for *any* solution, static or dynamic
- Pathlines not necessarily streamlines
- Circles with signed radius and frequency depending on n_3:
  $R(n_3) = \frac{-2k\sqrt{1-n_3^2}}{f'(n_3)}, \quad \omega(n_3) = |f'(n_3)|$

Example 1: Precessing uniform state

If we start with $\boldsymbol{n} \neq \boldsymbol{n}_0$ uniform everywhere, every pathline is a circle but they do not intersect at the same point in space and time. Precession with frequency $h + 2un_3$.

Example 2, axisymmetric breather

- When $u = 0$, all pathlines return to same point with period $\frac{2\pi}{h}$.
- Consider a contour of constant n_3, $\gamma = \frac{\pi}{2}$ with radius differing from $R(n_3)$ by $\Delta(n_3)$.
- The contour will oscillate above and below $R(n_3)$.
- While $\vec{\nu}$ tangent to its path-line, in general not tangent to the circle of constant n_3 - $\gamma(r, t)$ oscillating



Example 3, non-axisymmetric breather

$$\mathcal{R}(n_3, t) = \sqrt{R(n_3)^2 + \Delta(n_3)^2 + 2R(n_3)\Delta(n_3)\cos(ht - \Phi(n_3))}$$

- For our axisymmetric breathers, we already have an infinite-dimensional space of solutions given by $\Delta(n_3)$, $\Phi(n_3)$ (with Lipschitz constraints) - contrast isolated static axisymmetric solution.
- When we allow each oscillating circle of constant n_3 to have a different centre, introduce $\vec{X}_c(n_3)$ as before.
- Now $\vec{X}_c$, Δ and Φ all related through constraints

For finite-radius solutions, if $\Delta = 0$ at the boundary, this can be embedded in a uniform background as before.
Entire space of solutions, continuous or discontinuous, even more mysterious.

Future directions

Three dimensions

- The discussion extends straightforwardly to 3D, with the whole vector $\boldsymbol{n}$ rather than $\vec{\nu}$ playing the role of fluid velocity.
- General characteristics are almost all helices: no circles and no linking possible. So: no Hopfions.

Less symmetry, other generalisations

- tilted external field changes the characteristics to no longer be circular
- so does a spatially varying external field
- In each case, can characteristics be assembled to make static/dynamic solutions?

Future directions

Shocks

- So far we have carefully sidestepped discontinuous solutions
- We have to understand general shock conditions across discontinuities, and use them to construct more general dynamic (or static?) configurations

Infinitesimal Heisenberg interaction

- The space of skyrmion solutions can be thought of as a string with maximum local extension, but no tension
- An infinitesimal heisenberg term introduces tension, so that the axisymmetric skyrmion/meron is the true minimiser
- What about for breather solutions?

Summary

- A special limit of the chiral magnet model has infinite-dimensional families of static and dynamical solutions, which we can write down fairly explicitly.
- All solutions found so far are skyrmions or merons.
  - In 2D magnet we rule out continuous static antiskyrmions, skyrmionium etc.
  - In 3D magnet, we rule out continuous static Hopfion.
- Insights so far came from characteristics/ fluid dynamics intuition. Many future directions that can build on this idea, either looking for discontinuous solutions in the same model, or generalising e.g. potential term.
- Alternatively, for the solutions we have already, we can try to re-include infinitesimal Heisenberg interaction and see which static/dynamic solutions are selected - and thus find approximations to more physical solutions.