Interactions between solitons in chiral magnets: applications to soft and active matter

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Abstract

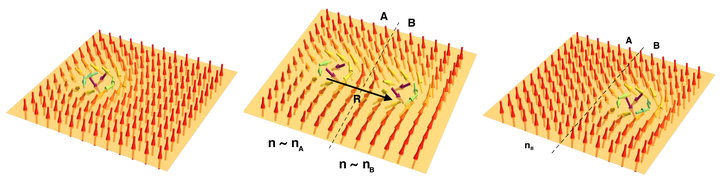
I review two methods of rigorously approximating interaction forces between localised textures in a continuum field theory. The first relies on the existence of an energy functional, and does not rely on a specific ansatz for superposing the soliton tails. It is calculated for the case of skyrmions in a chiral magnet in a tilted magnetic field background and compared to numerics. I also discuss its usefulness in studying liquid crystals.

The second method is a more general way to investigate effective dynamics of localised textures, which can be extended to systems not described by an energy functional. I discuss its application in studying antiferromagnetic skyrmion collapse, as well as how this generalisation allows study of active matter systems.

A general method for calculating interactions between exponentially localised solitons

Rough idea: **calculate energy of two textures at a given finite separation**, as a function of separation. (Below, we take example of magnetic textures.)

$$V_{int}(\vec{R}) = E(\mathbf{n}[\vec{R}]) - E(\mathbf{n}_A) - E(\mathbf{n}_B)$$



... but how do we define these configurations with fixed separation? They are by nature not static. But in fact, **it doesn't matter which superposition method we use**, provided:

- In the linearized regime (far from both textures), our configuration looks like a linear superposition of the tails of each isolated texture
- the perturbation of one texture on the core of the other is proportional to the size of the tail of the first texture
- the tails fall off exponentially

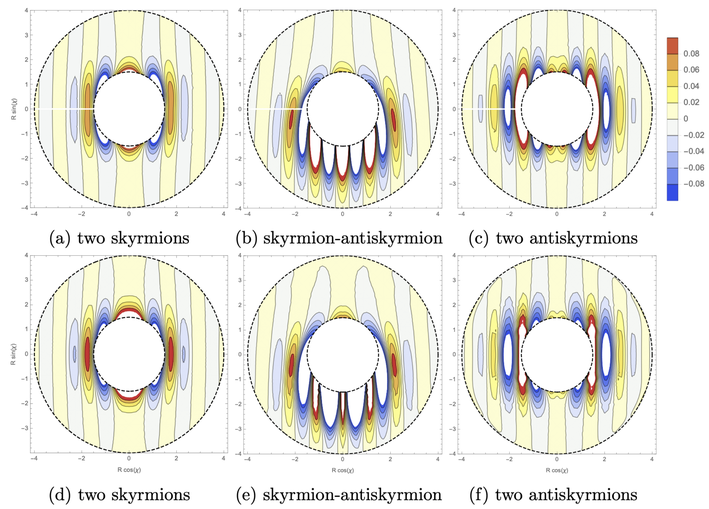
Under these assumptions, the leading interaction term as R becomes large is ansatz-independent and depends purely on the tails of the isolated solitons.[**?**, **?**, **?**]

Recipe:

1. Derive Euler-Lagrange equations of your energy, keeping track of boundary terms on the line separating the two textures
2. Take just the boundary term, and vary again to get bilinear form: $\int_{\partial\sigma^A} \epsilon(\ldots)\epsilon'$.
3. Find tails by solving linearised Euler-Lagrange equations
4. Plug tails into boundary integral, antisymmetrising: $V_{\text{int}} = \int_{\partial\sigma^A} \psi^A(\ldots)\psi^B - \int_{\partial\sigma^A} \psi^B(\ldots)\psi^A$.

Example: chiral skyrmions in tilted field: theory vs numerics

As a non-trivial test case, we can consider chiral skyrmions in thin-film magnets where an applied field is tilted with respect to the surface normal. The tilted uniform background means the DMI contributes to the interaction, unlike in standard chiral magnets. We can then compare the analytic form[1] we derive from the above method to numerical results in a previous paper [].



(a) two skyrmions (b) skyrmion-antiskyrmion (c) two antiskyrmions

(d) two skyrmions (e) skyrmion-antiskyrmion (f) two antiskyrmions

Example: the liquid crystal away from the one-constant approximation

One advantage of analytical formulae over numerics is the ability to explore a large range of the phase diagram quickly. With this in mind, we can apply our recipe to skyrmion textures interacting in the Frank-Oseen model of the liquid crystal, *without* applying the one-constant approximation.

$$E(\mathbf{n}) = \int \frac{1}{2}K_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_2\left(\mathbf{n}\cdot(\nabla\times\mathbf{n})\right)^2 + \frac{1}{2}K_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2$$

Applying the above recipe, we find the interaction energy in terms of the tails:

$$\begin{aligned} V_{\text{int}} =& K_1\int_\partial \left(\psi^A\nabla\cdot\psi^B - \psi^B\nabla\cdot\psi^A\right)\cdot d\mathbf{S} \\ &+K_2\int_\partial \left((\psi^A\times\mathbf{n}_0)\mathbf{n}_0\cdot(\nabla\times\psi^B) - (\psi^B\times\mathbf{n}_0)\mathbf{n}_0\cdot(\nabla\times\psi^A)\right)\cdot d\mathbf{S} \end{aligned}$$

Already, it is interesting to note that K_3, the 'bend' constant, does not play a direct role in the interaction energy (although it may indirectly, through its influence on the tail profiles.)

A general method for calculating dynamics/interactions in the absence of an energy

The above explicitly requires an energy: how can it apply to active matter, where there is typically no concept of an energy? In fact, a concept of interaction force can be derived from a totally different point of view, starting from the equations of motion.

The central idea is this: methods exist [**?**] for systematically expanding around a 'moduli space' of static solutions to a dynamical equation, in the adiabatic approximation where the fields are changing slowly. Typically, these rely on the linearisation of the equations around a static solution being self-adjoint - this is equivalent to the equations themselves coming from an energy. However, the method can be extended further.

Here, this generalisation is used to construct an 'artificial moduli space' of magnetic skyrmion profiles that are static solutions of *different* energy functionals.

Example of method (almost energy-free)

If we have some dynamical equations

$$\partial_t^2\Theta - \partial_r^2\Theta - \frac{1}{r}\partial_r\Theta + f(\Theta, r) = 0$$

where there is a family of static solutions parametrised by a free constant c, $\Theta(r,t) = \chi(r;c)$, then taking $\tau = \epsilon t$ and expanding $\Theta(r,t) = \Theta_0(r,t) + \epsilon^2\Theta_1(r,t)\ldots$ we find

$$\Theta_0(r,t) = \chi(r;c(\tau)), \tag{1}$$

i.e. to leading order, the dynamical solution adiabatically moves through static solutions. Moving to higher terms in the perturbative expansion, we find the corrections

$$L_\chi\Theta_i = \zeta(\Theta_{i-1}, \Theta_{i-2}\ldots\Theta_0), \tag{2}$$

where L_χ is the linearisation around the static solution. Since this equation comes from an energy functional, L_χ is self-adjoint and thus in particular $\langle\partial_c\chi, L_\chi\Theta_i\rangle = 0$. Thus the parallel part of the PDE gives us an effective ODE for $c(\tau)$ at each order, and the perpendicular part gives us $\Theta_i(r;c,\dot{c})$.

We can interpret this as saying that **the low-energy dynamics of the whole field theory is governed by the effective finite-d.o.f. dynamics of some localised configurations behaving 'like particles'**, and this ODE in general can have 'forces' that we can interpret as arising from potentials.

In particular, the equation above is used to investigate the dynamic of collapse of idealised skyrmions in a toy Heisenberg model.

Generalisation of the method (fully energy-free)

We want to extend this method to understand the collapse of a chiral antiferromagnetic skyrmion, described by the equation:

$$\partial_t^2\Theta - \left(\partial_r^2 + \frac{1}{r}\partial_r\right)\Theta + \frac{1}{2r^2}\sin(2\Theta) - \frac{2k}{r}\sin^2\Theta + h\sin(2\Theta) = 0. \tag{3}$$

These equations have a single static solution with a lengthscale fixed by k/h. But to study collapse we need a family with scale approaching zero.

So we have to construct an artificial, non-local equation:

$$\partial_t^2\Theta - \left(\partial_r^2 + \frac{1}{r}\partial_r\right)\Theta + \frac{1}{2r^2}\sin(2\Theta) - \frac{2\tilde{k}(\Theta)}{r}\sin^2\Theta + h\sin(2\Theta) = 0, \quad \tilde{k}(\Theta) = -\frac{1}{3}\Theta'(0) - \frac{2\Theta'''(0)}{3\Theta'(0)^2} + \frac{h}{\Theta'(0)} \tag{4}$$

such that static skyrmions from different theories with different k are stitched together in a single moduli space. The price we pay is that L_χ is not self-adjoint. $L_\chi\Theta_i$ is no longer perpendicular to $\partial_c\chi$, meaning the method has to be adjusted.

Non-self-adjoint L_χ will be the norm for active matter systems, so the fact that this method still works (even though it is applied to a passive matter system) is encouraging.

Summary

- There are two analytical ways to study interactions between localised textures in continuum field theories, energy-based and equations of motion-based.
- The first is ansatz-independent, for sufficiently localised textures, and shows good agreement with numerics. It gives simple formulae for studying energy functionals with many parameters.
- The second requires imagination to set up the ansatz, and truncating it at any order will give a force/dynamics depend on the precise ansatz chosen.
- The second method is generalisable to systems without an energy functional - eg non-reciprocal systems, active matter...

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[1] In fact, numerics are still required in this case to find the multipole sources associated to a single static skyrmion, which are undetermined constants in the analytical formula - crucially, this is numerically cheaper than direct calculation of inter-skyrmion potentials, and needs only be done once to find the interaction of the skyrmion with any other texture