Inter-skyrmion forces

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FOUNDATION FOR RESEARCH AND TECHNOLOGY - HELLAS

Work with Bernd Schroers, Stavros Komineas

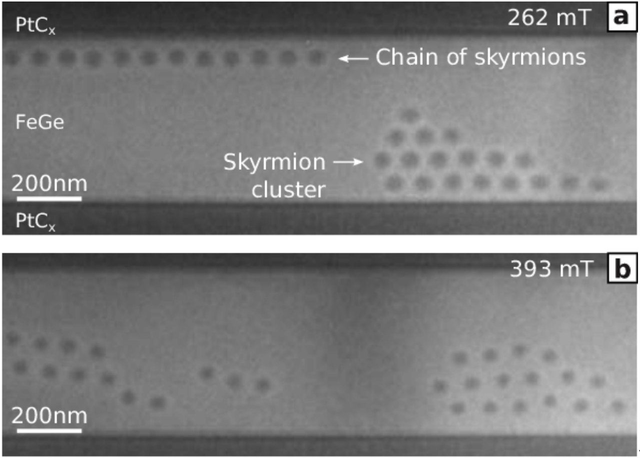
The headlines

Motivating question: Do topological textures repel or attract?

- If we know the profile of the 'tails' of two solitons in isolation, we can calculate the energy due to their interaction at long distance.
- The tails solve the linearised Euler-Lagrange equations so are easier to understand.
- Knowledge of the tails could come from analysis or numerics: either way, we have saved energy.

Magnetic skyrmions

Axisymmetric skyrmions in chiral magnets have been observed many times.

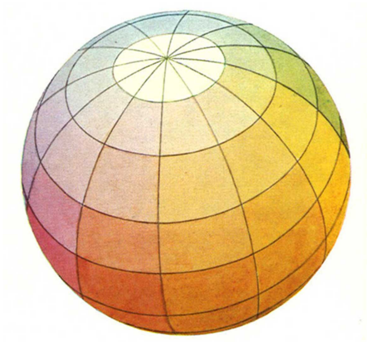


[1]

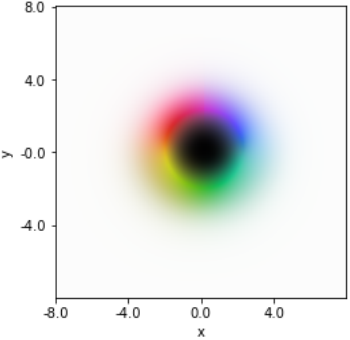
[1] Du et al. 2018.

Aside: colouring conventions

To represent configurations, we colour each point in the plane according to the direction of the magnetisation: black for down, white for up, ROYGBIV around the equator.



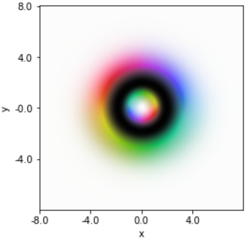
Runge Sphere



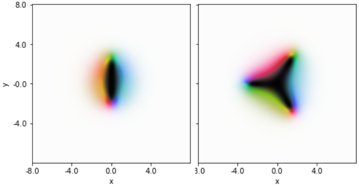
Skyrmion

Other magnetic solitons

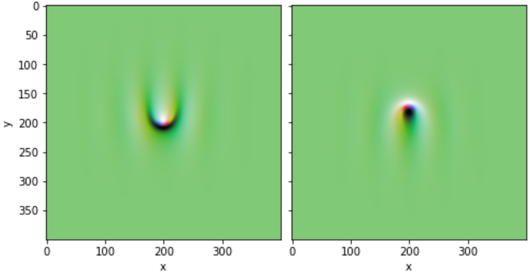
Skyrmionium



Non-axisymmetric solitons



Solitons in tilted field



Chiral magnet continuum model

$$E(\boldsymbol{n}) = \int \underbrace{\tfrac{1}{2}\partial_i \boldsymbol{n} \cdot \partial_i \boldsymbol{n}}_{\text{Heisenberg}} + \underbrace{\boldsymbol{A}_i \cdot (\boldsymbol{n} \times \partial_i \boldsymbol{n})}_{\text{DMI}} + \underbrace{h_z(1 - \boldsymbol{e}_h \cdot \boldsymbol{n})}_{\text{Zeeman}} + \underbrace{h_a(1 - n_3^2)}_{\text{anisotropy}} d^2x$$

Can be interpreted as $SU(2)$ gauge theory with covariant 'helical' derivative

$$D_i \boldsymbol{n} = \partial_i \boldsymbol{n} + \boldsymbol{A}_i \times \boldsymbol{n}$$

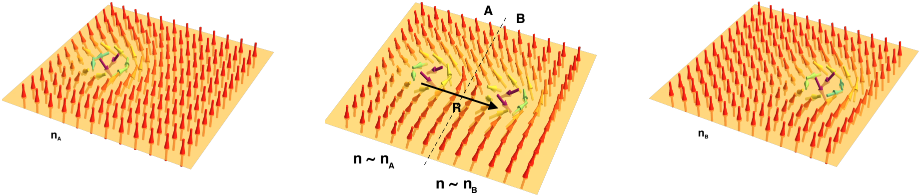
Gauge transformations $\boldsymbol{n} \mapsto \tilde{\boldsymbol{n}}$, $\boldsymbol{A}_i \mapsto \tilde{\boldsymbol{A}}_i$ relate different *models*

$\boldsymbol{A}_i = -kR(\alpha)\boldsymbol{e}_i$, 'axisymmetric DMI' is most commonly considered. If $\alpha = 0$: Bloch DMI $k\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n})$.

Calculating soliton interactions

Rough idea: calculate energy of configurations at finite separation, as a function of separation.

$$V_{int}(\vec{R}) = E(\boldsymbol{n}[\vec{R}]) - E(\boldsymbol{n}_A) - E(\boldsymbol{n}_B)$$



... but how do we define these configurations? They are by nature not static.

Constructing $\boldsymbol{n}$ as a function of $\vec{R}$:

- superposition ansatz
- $\boldsymbol{n}$ is minimal energy configuration subject to some constraint
- $\boldsymbol{n}$ achieved dynamically by some gradient flow

General result

It doesn't matter which method you choose! Provided that we assume:

- In the linearized regime (far from both skyrmions), our configuration looks like a linear superposition of the tails of each isolated soliton
- the perturbation of one soliton on the core of the other is proportional to the size of the tail of the first soliton
- the tails fall off exponentially

the leading term as R becomes large is ansatz-independent and depends purely on the tails of the isolated solitons.[2]

[2] Piette, Schroers, and Zakrzewski 1994; Kameda et al. 2021; Barton-Singer and Schroers 2023.

General result: sketch

Call the perturbation of soliton B on soliton A $\boldsymbol{\epsilon}^B = \exp^{-1}_{\boldsymbol{n}^A}(\boldsymbol{n})$
Call the tail of soliton B $\boldsymbol{\psi}^B = \exp^{-1}_{\boldsymbol{n}_0}(\boldsymbol{n}^B)$.
Then the interaction energy will be dominated by the first variation of the energy in one half-plane with respect to the perturbation coming from the other:

$$d_{\boldsymbol{n}} E_{\sigma^A}(\boldsymbol{\epsilon}) = \int_{\sigma^A} \boldsymbol{\epsilon} \cdot (\mathrm{EL}(\boldsymbol{n})) + \underbrace{\int_{\partial\sigma^A} f(\boldsymbol{\epsilon})}_{\alpha(\boldsymbol{\epsilon})}$$

and the bulk term will disappear since $\boldsymbol{n}$ is approximately a static solution, leaving the boundary term far from both solitons where, due to linear superposition, $\boldsymbol{\epsilon}^{A,B} \to \boldsymbol{\psi}^{A,B}$, (and both $\sim e^{-mr}$). Thus:

$$V_{int}(\vec{R}) \sim \underbrace{d_{\boldsymbol{n}_0} \alpha(\boldsymbol{\psi}_A, \boldsymbol{\psi}_B)}_{O(e^{-mR})} + O(e^{-\frac{3}{2}mR})$$

General result: the recipe

1. Derive Euler-Lagrange equations of your energy, keeping track of boundary terms
2. Take just the boundary term, and vary again to get bilinear form: $\int_{\partial\sigma^A} \boldsymbol{\epsilon}(\ldots)\boldsymbol{\epsilon}'$.
3. Find tails by solving linearised Euler-Lagrange equations
4. Plug tails into boundary integral, antisymmetrising: $V_{\text{int}} = \int_{\partial\sigma^A} \boldsymbol{\psi}^A(\ldots)\boldsymbol{\psi}^B - \int_{\partial\sigma^A} \boldsymbol{\psi}^B(\ldots)\boldsymbol{\psi}^A$.

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We will now apply this formula to various scenarios in the chiral magnet.

Result for the general chiral magnet model

$$E(\boldsymbol{n}) = \int \frac{1}{2}\partial_i \boldsymbol{n} \cdot \partial_i \boldsymbol{n} + \boldsymbol{A}_i \cdot (\boldsymbol{n} \times \partial_i \boldsymbol{n}) + h_z(1 - \boldsymbol{e}_h \cdot \boldsymbol{n}) + h_a(1 - n_3^2) d^2x$$

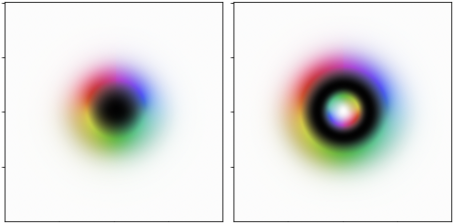
Introducing the complex field $\psi = \boldsymbol{\psi}_1 + i\boldsymbol{\psi}_2$, and $a_i = \boldsymbol{A}_i \cdot \boldsymbol{n}_0$

$$V_{\text{int}}(\vec{R}) = 2Re \int_{\partial A} (\bar{\psi}_B(\vec{\partial} - i\vec{a})\psi_A) \cdot d\vec{S} + O(e^{-\frac{3}{2}mR})$$

- Potential terms do not (directly) contribute.
- $\boldsymbol{A}_i$ generally lie in plane (even for relatively low symmetry), and typically $\boldsymbol{n}_0 \parallel \boldsymbol{e}_3$, so the DMI, too, may not feature.

Axisymetric solitons

Simplest case: our energy is rotationally symmetric ($\boldsymbol{e}_h = \boldsymbol{e}_3$, $\boldsymbol{A}_i = -k\boldsymbol{e}_i \implies \vec{a} = \vec{0}$) and we look for solitons that keep this symmetry.



By symmetry, individual tail has form $\psi = iq_1 e^{i\phi} K_1(mr)$ (dipole source, fixed orientation), where $m = \sqrt{h_z + 2h_a}$.

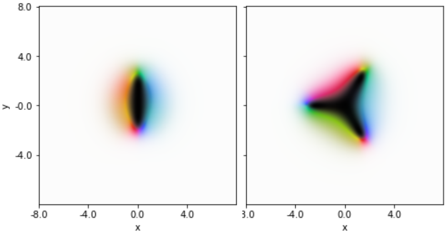
$$V_{\text{int}}(\vec{R}) \sim 2Re\left(\int_{\partial\sigma_A} (\bar{\psi}^B \vec{\partial}\psi^A) \cdot d\vec{S}\right) \propto q_1^A q_1^B K_0(mR),$$

and thus they always repel.[3]

[3]Foster et al. 2019.

Non-axisymmetric skyrmions

Many non-axisymmetric skyrmions are possible where chirality reverses in localised 'kinks'. Since they spontaneously break axisymmetry, they have a rotational zero mode.



For *some orientations*, these non-axisymmetric skyrmions attract both axisymmetric skyrmions and each other.

- Tails: in general, $\psi = \sum_M q_M e^{iM\phi + i\gamma_M} K_M(mr)$, with γ_n fixed by symmetry as function of orientation.
- But most importantly for us, ψ does not attain 0 in linear regime: winding around core of soliton is preserved far away.

Non-axisymmetric solitons part 2

We can substitute the general multipole expansion into the interaction energy:

$$V_{\text{int}}(R, \chi) \sim \sum_{M,N} \# q_M^A q_N^B \cos(\gamma_M^A - \gamma_N^B + (M - N)\chi) K_{|M-N|}(mR)$$

Already this allows us to predict interaction quantitatively using numerically found multipole sources of antiskyrmion, etc.
But we can go further: in $1/x$ expansion for $K_n(x)$, we see leading interaction is just inner product of tails at midpoint between skyrmions:

$$V_{\text{int}}(\vec{R}) = -\boldsymbol{\psi}^A(R, \chi) \cdot \boldsymbol{\psi}^B(R, \chi + \pi) e^{mR} \sqrt{mR} + O\left(\frac{e^{-mR}}{(mR)^{\frac{3}{2}}}\right)$$

Because each $\boldsymbol{\psi}$ winds around $\boldsymbol{e}_3$ as we vary orientation, we can make this interaction negative: in general, non-axisymmetric solitons repel or attract depending on orientation.

Bilayer skyrmions

Two chiral magnet layers with antiferromagnetic interaction between them:

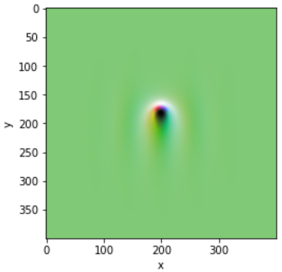
$$E(\boldsymbol{n}^U, \boldsymbol{n}^L) = E(\boldsymbol{n}^U) + E(\boldsymbol{n}^L) + j_c \int \boldsymbol{n}^U \cdot \boldsymbol{n}^L d^2x$$

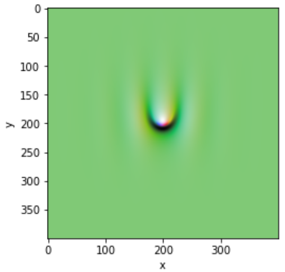
Bilayer skyrmions and monolayer skyrmions both stable solutions
Monolayer skyrmion has corresponding topologically trivial 'shadow' on other layer, with its own tail.

Result

- Dipole charge of monolayer skyrmion and shadow are equal.
- All solutions repel, with same profile independent of j_c.
- In particular, two monolayer skyrmions on opposite layers have energy barrier to combining into a bilayer skyrmion.
- At small j_c, two monolayer skyrmions have same interaction energy when on same or opposite layers

Tilted field: numerical comparison





Overlap of DMI and $\boldsymbol{n}_0$ puts oscillation on top of tail:

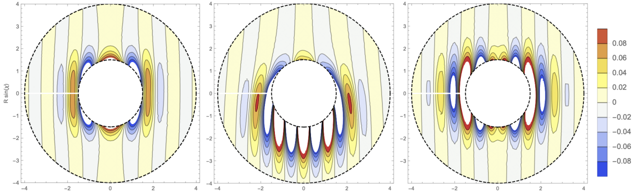
$$\psi = e^{-i\vec{a}\cdot\vec{r}}\tilde{\psi},\ \tilde{\psi} = \sum_M q_M e^{iM\phi + i\gamma_M} K_M(mr),\ m = \sqrt{h_z^2 - |\vec{a}|^2}$$

We can find q_M numerically.
The interaction energy is

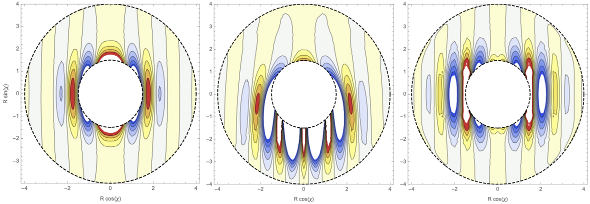
$$V_{\text{int}}(\vec{R}) = 2Re\left(e^{-i\vec{a}\cdot\vec{R}} \int_{\partial\sigma_A} (\bar{\tilde{\psi}}^B \vec{\partial}\tilde{\psi}^A)\cdot d\vec{S}\right) + O(e^{-\frac{3}{2}mR})$$

Tilted field: numerical comparison

$$V_{\text{int}}(R,\chi) = \sum_{M,N} \frac{(-1)^{N+1}}{2\pi} m^{|M|+|N|} q^A_M q^B_N \cos(\gamma^A_M - \gamma^B_N - aR\cos\chi + (M-N)\chi) K_{|M-N|}(mR)$$



(a) two skyrmions (b) skyrmion-antiskyrmion (c) two antiskyrmions



(d) two skyrmions (e) skyrmion-antiskyrmion (f) two antiskyrmions

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[4] Kuchkin and Kiselev 2020; Barton-Singer and Schroers 2023.

Bonus: Colliding skyrmions in antiferromagnets[5]

- Antiferromagnets have same static energy functional for Néel vector, but dynamics is $(\ddot{\boldsymbol{n}})_\perp = -\left(\frac{\delta E}{\delta \boldsymbol{n}}\right)_\perp$
- This allows for freely propagating skyrmions. Question: what happens when two propagating skyrmions collide?
- So far, discussing 'repelling' or 'attracting' implicitly assume that the dynamics only involves $\vec{R}$ varying as a function of time.
- However, a positive interaction energy could also be lowered by lowering q_1, which is linked to skyrmion size. (Explicit formula at small size.)

Thus:

$$\ddot{\vec{R}} = -\frac{\partial V_{\text{int}}}{\partial \vec{R}}, \qquad \ddot{q}_1^{A,B} = -\frac{\partial V_{\text{int}}}{\partial q_1^{A,B}} - E'_{Sk}(q_1^{A,B})$$

Collision excites breathing dynamics.

[5] Theodorou, Barton-Singer, and Komineas 2025.

Summary

- We described a recipe for calculating the leading term of inter-soliton forces - ansatz-independent so reflects something fundamental.
- We see the results this gives for standard Bloch skyrmions as well as more exotic cases.
- Analytical predictions give good agreement with numerics.
- Even when numerical simulation is needed to fix constants in interaction term, we only need to simulate each isolated soliton once.
- Attraction or repulsion is not (directly) related to topological charge.
- Just because two skyrmions would annihilate or combine to form something with lower energy, their interactions need not be attractive.
- A polarised background tilted so that it overlaps with the DMI vectors gives rise to oscillating interactions.